Problem 13) Using the Taylor series $\sum_{n=0}^{\infty}(x^n/n!)$ for the function e^x, we write

$$(1+x)e^x = (1+x)\sum_{n=0}^{\infty}(x^n/n!) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots\right)$$
$$+\left(x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots + \frac{x^{n+1}}{n!} + \cdots\right)$$
$$= 1 + (1+1)x + \left(1 + \frac{1}{2!}\right)x^2 + \left(\frac{1}{2!} + \frac{1}{3!}\right)x^3 + \cdots + \left[\frac{1}{(n-1)!} + \frac{1}{n!}\right]x^n + \cdots$$
$$= 1 + \frac{2}{1!}x + \frac{3}{2!}x^2 + \frac{4}{3!}x^4 + \cdots + \frac{(n+1)}{n!}x^n + \cdots$$
$$= \sum_{n=0}^{\infty}(n+1)x^n/n!.$$

Alternatively, one could systematically go about computing the various derivatives of the function $f(x) = (1+x)e^x$, as follows:

$$f'(x) = (2+x)e^x \quad \rightarrow \quad f'(0) = 2,$$
$$f''(x) = (3+x)e^x \quad \rightarrow \quad f''(0) = 3,$$
$$f'''(x) = (4+x)e^x \quad \rightarrow \quad f'''(0) = 4,$$
$$\vdots$$
$$f^{(n)}(x) = (n+1+x)e^x \quad \rightarrow \quad f^{(n)}(0) = n+1.$$

Consequently, $f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)x^n/n! = \sum_{n=0}^{\infty}(n+1)x^n/n!$.